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Constraining maximally supersymmetric membrane actions

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Abstract: We study the recent construction of maximally supersymmetric field theory Lagrangians in three spacetime dimensions that are based on algebras with a triple product. Assuming that the algebra has a positive definite metric compatible with the triple product, we prove that the only non-trivial examples are either the well known case based on a four dimensional algebra or direct sums thereof.

Keywords: Supersymmetric gauge theory, M-Theory.

Contents

1. Introduction

A better understanding of the three-dimensional superconformal field theory that arises on multiple membranes in flat space is an important outstanding issue in M-theory. Building on earlier work [1, 2], an interesting Lagrangian description of a maximally supersymmetric conformal field theory in three dimensions was constructed in [3 − 5] which has been further studied in [6]–[19]. The construction relies on an algebra with a skew triple product whose structure constants $f^{\mu_1\mu_2\mu_3}{}_\nu = f^{[\mu_1\mu_2\mu_3]}{}_\nu$ satisfy

$$f^{\mu_1\mu_2\mu_3}{}_\nu f^{\mu_4\mu_5\nu}{}_{\mu_6} = 3 f^{\mu_4\mu_5[\mu_1}{}_\nu f^{\mu_2\mu_3]\nu}{}_{\mu_6} \tag{1.1}$$

or equivalently

$$f^{[\mu_1\mu_2\mu_3}{}_\nu f^{\mu_4]\mu_5\nu}{}_{\mu_6} = 0 \ . \tag{1.2}$$

The construction of the Lagrangian requires a compatible metric and, after raising an index on f using this metric, f is totally antisymmetric $f^{\mu_1\mu_2\mu_3\mu_4} = f^{[\mu_1\mu_2\mu_3\mu_4]}$. Since the metric appears in the kinetic terms of the Lagrangian, it is natural to demand that the metric is positive definite. In this case, after a suitable change of basis, we can assume that the metric is simply $\delta_{\mu\nu}$. The basic non-trivial solution [5] corresponds to a four dimensional algebra with $f^{\mu_1\mu_2\mu_3\mu_4} = \epsilon^{\mu_1\mu_2\mu_3\mu_4}$. One can also consider direct sums of this basic example, but this simply leads to three-dimensional supersymmetric field theories which are non-interacting copies of the basic example.

We started this work by trying to construct additional solutions to (1.2) with totally antisymmetric f. However, as also noticed by others, obvious generalisations fail and simple computer searches are fruitless. It has also been shown [20] that in up to seven dimensions, a 4-form whose components satisfy (1.2) must be proportional to dx^{1234} (in some appropriately chosen co-ordinates), and in eight dimensions, the solution is a linear combination dx^{1234} and dx^{5678}.

Here we will prove the general result, that all solutions of (1.2), in any dimension, can be written as a linear combination 4-forms, each of which is the wedge product of four 1-forms, which are all mutually orthogonal. This then proves conjectures made in [20] and [16].

Note added. Concurrent with the posting of this work to the ArXive, a proof of this result also appeared in [21]. After this paper was submitted for publication, we became aware of [22], which claims the same result using a different approach.

2. Analysis

We are interested in solutions to (1.2) for totally anti-symmetric and real f with indices raised and lowered using the metric $\delta_{\mu\nu}$. Let us assume that we have a $D+1$ dimensional algebra and write the indices as $\mu = (q, D+1)$ where $q = 1, \dots, D$. We can write

$$f = dx^{D+1} \wedge \psi + \phi \tag{2.1}$$

where ψ is a 3-form on $\mathbb{R}^D$, and ϕ is a 4-form on $\mathbb{R}^D$. We can demand that $\psi \neq 0$ (otherwise we end up in D dimensions). The constraint (1.2) is equivalent to

$$\phi^{[q_1 q_2 q_3}{}_m \phi^{q_4]q_5 q_6 m} + \psi^{[q_1 q_2 q_3} \psi^{q_4]q_5 q_6} = 0 \tag{2.2}$$

$$q\phi^{[q_1 q_2 q_3}{}_m \psi^{q_4]q_5 m} = 0 \tag{2.3}$$

$$\phi^{q_1 q_2 q_3}{}_m \psi^{q_4 q_5 m} - 3\psi^{[q_1 q_2}{}_m \phi^{q_3]q_4 q_5 m} = 0 \tag{2.4}$$

$$\psi^{[q_1 q_2}{}_m \psi^{q_3]q_4 m} = 0 \tag{2.5}$$

where indices on ψ, ϕ are raised/lowered with δ_{mn}. Observe that (2.5) is the Jacobi identity. This identity implies that $\psi_{mn}{}^p$ are the structure constants of a Lie algebra $\mathcal{L}$. The Killing form of this Lie algebra has components

$$\kappa_{mn} = \psi_{m\ell}{}^p \psi_{np}{}^\ell \ . \tag{2.6}$$

As ψ is totally antisymmetric, note that κ is negative semi-definite. There are two possibilities: κ is non-degenerate and $\mathcal{L}$ is semi-simple or κ is degenerate.

Suppose that $\mathcal{L}$ is semi-simple. By making a SO(D) rotation, one can diagonalize the Killing form and set

$$\kappa_{mn} = -\lambda_n \delta_{mn} \tag{2.7}$$

(no sum over n), and $\lambda_n > 0$ for all n.

On the other hand if κ is degenerate, then $\mathcal{L} = u(1)^p \oplus \mathcal{L}'$ where $p > 0$ and $\mathcal{L}$ is semi-simple. To see this we first note that $X^m \kappa_{mn} = 0$ for some non-zero vector X^n. Then it follows that

$$X^m X^n \psi_{mpq} \psi_n{}^{pq} = 0 \tag{2.8}$$

which implies that $X^n \psi_{npq} = 0$. Without loss of generality, one can make an SO(D) rotation so that the only non-vanishing component of X^n is X^1 and then $\psi_{1mn} = 0$ for all m, n, and $\kappa_{1m} = 0$ for all m. By repeating this process in the directions $2, \dots, D$ one finds after a finite number of steps, either that $\mathcal{L} = u(1)^p \oplus \mathcal{L}'$ where $p > 0$ and $\mathcal{L}'$ is semi-simple, or $\psi = 0$ which we have assumed not to be the case.

We will analyse the two cases in turn, but we first establish some useful identities arising from (2.3)–(2.5) that are valid in both cases. We define $h = -\kappa$ i.e.

$$h_{mn} = \psi_{mab} \psi_n{}^{ab} \ . \tag{2.9}$$

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First contract (2.3) with $\psi_{q_4q_5\ell}$ so that one obtains

$$\phi^{q_1q_2q_3m}h_{m\ell}-\phi^{q_4q_2q_3m}\psi^{q_5q_1}{}_m\psi_{q_5q_4\ell}-\phi^{q_1q_4q_3m}\psi^{q_5q_2}{}_m\psi_{q_5q_4\ell}-\phi^{q_1q_2q_4m}\psi^{q_5q_3}{}_m\psi_{q_5q_4\ell}=0\ . \tag{2.10}$$

However, note that the Jacobi identity implies that

$$\phi^{q_4q_2q_3m}\psi^{q_5q_1}{}_m\psi_{q_5q_4\ell}=\frac{1}{2}\phi^{q_2q_3mn}\psi^{rq_1}{}_\ell\psi_{rmn}\ . \tag{2.11}$$

Using this identity one can rewrite (2.10) as

$$\phi^{q_1q_2q_3m}h_{m\ell}-\frac{1}{2}\phi^{q_2q_3mn}\psi^{rq_1}{}_\ell\psi_{rmn}-\frac{1}{2}\phi^{q_3q_1mn}\psi^{rq_2}{}_\ell\psi_{rmn}-\frac{1}{2}\phi^{q_1q_2mn}\psi^{rq_3}{}_\ell\psi_{rmn}=0\ . \tag{2.12}$$

Also, contracting (2.3) with $\delta_{q_3q_5}$ gives

$$\phi^{q_1q_2mn}\psi^{q_4}{}_{mn}+\phi^{q_2q_4mn}\psi^{q_1}{}_{mn}+\phi^{q_4q_1mn}\psi^{q_2}{}_{mn}=0\ . \tag{2.13}$$

Next, contract (2.4) with $\psi_{q_1q_2\ell}$ to obtain

$$-\phi^{q_3q_4q_5m}h_{m\ell}+\phi^{q_1q_2q_3m}\psi_{q_1q_2\ell}\psi^{q_4q_5}{}_m-2\phi^{q_2q_4q_5m}\psi^{q_3q_1}{}_m\psi_{q_1q_2\ell}=0\ . \tag{2.14}$$

This can be rewritten (using (2.11) to simplify the last term) as

$$-\phi^{q_1q_2q_3m}h_{m\ell}+\phi^{mnq_1r}\psi_{mn\ell}\psi^{q_2q_3r}+\phi^{q_2q_3mn}\psi^{rq_1}{}_\ell\psi_{rmn}=0\ . \tag{2.15}$$

On contracting this expression with $\delta_{q_1q_3}$, the first and the third term vanish (the third term vanishes as a consequence of the Jacobi identity), and we find

$$\phi^{n_1n_2m_1m_2}\psi_{n_1n_2\ell}\psi_{m_1m_2r}=0\ . \tag{2.16}$$

Next, contract (2.15) with $\psi_{q_2q_3s}$. The last term vanishes as a consequence of (2.16), and we obtain

$$-\phi^{mnqr}h_{r\ell}\psi_{mns}+\phi^{mnqr}h_{rs}\psi_{mn\ell}=0\ . \tag{2.17}$$

2.1 Solutions when $\mathcal{L}$ is semi-simple

We now assume that $\mathcal{L}$ is semi-simple. As we have already observed, we can make a rotation and work in a basis for which

$$h_{mn}=\lambda_n\delta_{mn} \tag{2.18}$$

(no sum over n), with $\lambda_n>0$ for all n.

Then (2.17) implies

$$-\phi^{mnq}{}_\ell\lambda_\ell\psi_{mns}+\phi^{mnq}{}_s\lambda_s\psi_{mn\ell}=0 \tag{2.19}$$

with no sum over ℓ or s. On substituting this expression back into (2.13) we obtain

$$(\lambda_{q_4}-\lambda_{q_1}-\lambda_{q_2})\phi^{q_1q_2mn}\psi^{q_4}{}_{mn}=0 \tag{2.20}$$

(no sum on q_1,q_2,q_4). Hence $\phi^{q_1q_2mn}\psi^{q_4}{}_{mn}=0$, or $\lambda_{q_4}-\lambda_{q_1}-\lambda_{q_2}=0$ for some choice of q_1,q_2,q_4. Now, it is not possible to have $\lambda_{q_4}-\lambda_{q_1}-\lambda_{q_2}=\lambda_{q_1}-\lambda_{q_2}-\lambda_{q_4}=\lambda_{q_2}-\lambda_{q_1}-\lambda_{q_4}=0$

simultaneously. Hence, at least one of $\phi^{q_1q_2mn}\psi^{q_4}{}_{mn}$, $\phi^{q_1q_4mn}\psi^{q_2}{}_{mn}$, $\phi^{q_2q_4mn}\psi^{q_1}{}_{mn}$ must vanish. However, (2.19) then implies that *all* these terms vanish. Hence we conclude that

$$\phi^{q_1q_2mn}\psi^{q_4}{}_{mn} = 0 \tag{2.21}$$

for all q_1, q_2, q_4. Finally, on substituting (2.21) back into (2.12), the last three terms are constrained to vanish, hence

$$\phi_{q_1q_2q_3q_4} = 0 \ . \tag{2.22}$$

Now consider (2.2). This implies that

$$\psi^{[q_1q_2q_3}\psi^{q_4]q_5q_6} = 0 \tag{2.23}$$

which implies (see e.g. [20]) that ψ is simple i.e. it can be written as the wedge product of three one forms. Hence one can chose a basis for which

$$\psi = \lambda dx^1 \wedge dx^2 \wedge dx^3 \ . \tag{2.24}$$

Furthermore, as $\mathcal{L}$ is compact, this implies that $\mathcal{L}$ must be 3-dimensional i.e. $\mathcal{L} = su(2)$. We have thus recovered the basic four-dimensional case with $f^{\mu_1\mu_2\mu_3\mu_4} = \epsilon^{\mu_1\mu_2\mu_3\mu_4}$.

2.2 Solutions when $\mathcal{L}$ is not semi-simple

Set $\mathcal{L} = u(1)^p \oplus \mathcal{L}'$ where $p > 0$ and $\mathcal{L}'$ is semi-simple. It will be useful to split the indices m into "semi-simple" directions $\hat{m}$ and "$u(1)$" directions A, so $m = (\hat{m}, A)$. Note that $\psi_{Amn} = 0$ for all m, n, and $h_{Am} = 0$ for all m, but $h_{\hat{m}\hat{n}} = \lambda_{\hat{n}}\delta_{\hat{m}\hat{n}}$ (no sum on $\hat{n}$). Recall the identity (2.12). Setting $q_1 = A$, $q_2 = B$, $q_3 = C$ one finds

$$\phi_{ABC\hat{m}} = 0 \ . \tag{2.25}$$

Also, setting $q_1 = A$, $q_2 = B$, $q_3 = \hat{m}$ one finds

$$\phi^{AB\hat{m}\hat{s}}h_{\hat{s}\hat{\ell}} - \frac{1}{2}\phi^{AB\hat{p}\hat{q}}\psi^{\hat{s}\hat{m}}{}_{\hat{\ell}}\psi_{\hat{s}\hat{p}\hat{q}} = 0 \ . \tag{2.26}$$

However, (2.13) implies that

$$\phi^{AB\hat{p}\hat{q}}\psi_{\hat{s}\hat{p}\hat{q}} = 0 \tag{2.27}$$

and so on substituting this back into (2.26) one finds

$$\phi_{AB\hat{m}\hat{n}} = 0 \ . \tag{2.28}$$

Returning to the general conditions (2.3), (2.4) and (2.5) with all free indices hatted, we can follow the same steps in the last subsection to conclude that

$$\phi_{\hat{m}\hat{n}\hat{p}\hat{q}} = 0 \ . \tag{2.29}$$

Thus the only non-zero components of ϕ are of the form $\phi^{A\hat{q}_1\hat{q}_2\hat{q}_3}$ and ϕ^{ABCD}.

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Considering other indices in (2.3), (2.4) and (2.5) we conclude that

$$\psi^{[\hat{q}_1\hat{q}_2}{}_{\hat{m}}\psi^{\hat{q}_3]\hat{q}_4\hat{m}} = 0 \tag{2.30}$$

$$\phi^{A\hat{q}_1\hat{q}_2}{}_{\hat{m}}\psi^{\hat{q}_3\hat{q}_4\hat{m}} = \phi^{A\hat{q}_3\hat{q}_4}{}_{\hat{m}}\psi^{\hat{q}_1\hat{q}_2\hat{m}} \tag{2.31}$$

$$\phi^{A\hat{q}_1[\hat{q}_2}{}_{\hat{m}}\psi^{\hat{q}_3\hat{q}_4]\hat{m}} = 0\ . \tag{2.32}$$

From (2.2) we also get

$$\phi^{[A_1A_2A_3}{}_{B}\phi^{A_4]A_5A_6B} = 0 \tag{2.33}$$

$$\phi^{\hat{q}_1\hat{q}_2\hat{q}_3}{}_{B}\phi^{A_1A_2A_3B} = 0 \tag{2.34}$$

$$\phi^{A[\hat{q}_1\hat{q}_2}{}_{\hat{m}}\phi^{\hat{q}_3]\hat{q}_4B\hat{m}} = 0 \tag{2.35}$$

$$\phi^{\hat{q}_1\hat{q}_2}{}_{\hat{m}[A_1}\phi_{A_2]}{}^{\hat{q}_3\hat{q}_4\hat{m}} = 0 \tag{2.36}$$

$$\psi^{[\hat{q}_1\hat{q}_2\hat{q}_3}\psi^{\hat{q}_4]\hat{q}_5\hat{q}_6} + \phi^{[\hat{q}_1\hat{q}_2\hat{q}_3}{}_{A}\phi^{\hat{q}_4]\hat{q}_5\hat{q}_6A} = 0\ . \tag{2.37}$$

To proceed with the analysis, it is convenient to define the matrices T^A by

$$(T^A)_{\hat{m}}{}^{\hat{n}} = \phi^{A\hat{q}_1\hat{q}_2\hat{n}}\psi_{\hat{q}_1\hat{q}_2\hat{m}}\ . \tag{2.38}$$

On contracting (2.31) with $\delta_{\hat{q}_2\hat{q}_4}$, we observe that T^A are all symmetric matrices. Furthermore, on contracting (2.36) with $\delta_{\hat{q}_2\hat{q}_4}$ and making use of (2.31), it is straightforward to show that the matrices T^A commute with each other. Also, (2.31) implies that the T^A commute with h.

Next, note that the Jacobi identity (2.30) implies that

$$(T^A)_{\hat{m}\hat{\ell}}\psi^{\hat{\ell}}{}_{\hat{p}\hat{q}} = \phi^{A\hat{s}\hat{t}}{}_{\hat{m}}\psi_{\hat{s}\hat{t}\hat{\ell}}\psi^{\hat{\ell}}{}_{\hat{p}\hat{q}} = -2\phi^{A\hat{s}\hat{t}}{}_{\hat{m}}\psi_{\hat{s}\hat{p}\hat{\ell}}\psi^{\hat{\ell}}{}_{\hat{q}\hat{t}} \tag{2.39}$$

However, now using (2.31) and then the Jacobi identity again, we get

$$-2\phi^{A\hat{s}\hat{t}}{}_{\hat{m}}\psi_{\hat{s}\hat{p}\hat{\ell}}\psi^{\hat{\ell}}{}_{\hat{q}\hat{t}} = -2\phi^{A\hat{s}}{}_{\hat{p}\hat{l}}\psi_{\hat{s}}{}^{\hat{t}}{}_{\hat{m}}\psi^{\hat{l}}{}_{\hat{q}\hat{t}} = -\phi^{A\hat{s}\hat{l}}{}_{\hat{p}}\psi_{\hat{s}\hat{l}\hat{t}}\psi^{\hat{t}}{}_{\hat{m}\hat{q}}\ . \tag{2.40}$$

Thus

$$(T^A)_{\hat{m}\hat{\ell}}\psi^{\hat{\ell}}{}_{\hat{p}\hat{q}} = -(T^A)_{\hat{p}\hat{\ell}}\psi^{\hat{\ell}}{}_{\hat{m}\hat{q}}\ . \tag{2.41}$$

Next, decompose semi-simple $\mathcal{L}' = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_m$ where $\mathcal{L}_i$ are simple ideals such that $\mathcal{L}_i \perp \mathcal{L}_j$ (with respect to h), and $[\mathcal{L}_i, \mathcal{L}_j] = 0$ if $i \neq j$, and the restriction of the adjoint rep. to $\mathcal{L}_i$ is irreducible; furthermore, $h|_{\mathcal{L}_i} = 2\mu_i^2\mathbb{I}$ for $\mu_i \neq 0$. Contract (2.31) with $\psi_{\hat{q}_3\hat{q}_4\hat{\ell}}$ to obtain

$$\phi^{A\hat{q}_1\hat{q}_2\hat{m}}h_{\hat{m}\hat{\ell}} = \phi^{A\hat{q}_3\hat{q}_4\hat{m}}\psi^{\hat{q}_1\hat{q}_2}{}_{\hat{m}}\psi_{\hat{q}_3\hat{q}_4\hat{\ell}}\ . \tag{2.42}$$

Suppose that the indices $\hat{q}_1, \hat{q}_2$ lie in two different ideals $\mathcal{L}_i, \mathcal{L}_j$ for $i \neq j$. Then the r.h.s. of the above expression vanishes, hence for these indices, $\phi_{A\hat{q}_1\hat{q}_2\hat{m}} = 0$, for all $\hat{m}$. Similarly, for these indices $(T^A)_{\hat{q}_1}{}^{\hat{q}_2} = \phi^{A\hat{r}\hat{\ell}\hat{q}_2}\psi_{\hat{r}\hat{\ell}\hat{q}_1} = 0$.

Consider T_i^A, the restriction of T^A to $\mathcal{L}_i$. Then (2.41) implies that T_i^A commutes with the restriction of the adjoint rep. to $\mathcal{L}_i$. However, as this restriction of the adjoint rep. is irreducible, it follows by Schur's Lemma that

$$T_i^A = \lambda_i^A\mathbb{I}\ . \tag{2.43}$$

As the T^A all commute, this can be achieved for all T^A.

Next, consider (2.37) with all $\hat{q}$ indices restricted to $\mathcal{L}_i$. Contracting this expression with $\psi_{\hat{q}_1\hat{q}_2\hat{q}_3}\psi_{\hat{q}_5\hat{q}_6\hat{m}}$ gives

$$\left(\sum_A (\lambda_i^A)^2 + 4(\mu_i)^4\right)(\dim \mathcal{L}_i - 3)\,\delta^{\hat{q}_4}_{\hat{m}} = 0 \tag{2.44}$$

which implies that dim $\mathcal{L}_i = 3$ for all i, so $\mathcal{L}_i = su(2)$. It follows that

$$\psi = \sum_i \mu_i \theta_i \tag{2.45}$$

with $\mu_i \neq 0$, where

$$\theta_i = dy_i^1 \wedge dy_i^2 \wedge dy_i^3 \tag{2.46}$$

If the $\hat{q}$ indices are restricted to $\mathcal{L}_i$, since dim $\mathcal{L}_i = 3$, $\phi_{A\hat{q}_1\hat{q}_2\hat{q}_3}$ must be proportional to θ_i. The proportionality constant can be fixed from (2.43) and we find

$$\phi_{A\hat{q}_1\hat{q}_2\hat{q}_3} = \frac{\lambda_i^A}{2\mu_i}(\theta_i)_{\hat{q}_1\hat{q}_2\hat{q}_3}\ . \tag{2.47}$$

It is convenient to re-define $\lambda_i^A = 2\mu_i\chi_i^A$, so that

$$f = dx^{d+1} \wedge \psi + \sum_{i,A} \chi_i^A dz^A \wedge \theta_i + \Phi \tag{2.48}$$

where Φ lies entirely in the $u(1)$ directions, whose directions we have denoted by z^A. The remaining content of (2.37) is obtained by restricting the indices $\hat{q}_1, \hat{q}_2, \hat{q}_3$ to $\mathcal{L}_i$, and $\hat{q}_4, \hat{q}_5, \hat{q}_6$ to $\mathcal{L}_j$ for $i \neq j$; we find

$$\mu_i\mu_j + \sum_A \chi_i^A \chi_j^A = 0\ . \tag{2.49}$$

Note that the form Φ satisfies the quadratic constraint (2.33), whereas (2.34) is equivalent to

$$\chi_i^A \Phi_{AMNP} = 0 \tag{2.50}$$

for all i.

There are then two cases to consider. In the first case, $\chi_i^A = 0$ for all A, i. Then (2.49) implies that $\mathcal{L}' = su(2)$, and hence

$$f = \mu_1 dx^{d+1} \wedge dy_1^1 \wedge dy_1^2 \wedge dy_1^3 + \Phi \tag{2.51}$$

where Φ has no components in the $x^{d+1}, y_1^1, y_1^2, y_1^3$ directions.

In the second case, there exists some A, i with $\chi_i^A \neq 0$. Without loss of generality, take $i = 1$. By making an SO(p) rotation entirely in the $u(1)$ directions, without loss of generality set

$$\chi_1^1 = \tau, \qquad \chi_1^A = 0 \quad \text{if A} > 1 \tag{2.52}$$

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where $\tau \neq 0$. Then, if $j \neq 1$, (2.49) implies that

$$\chi_j^1 = -\frac{\mu_1}{\tau}\mu_j \ . \tag{2.53}$$

Substituting these constraints back into (2.48), and rearranging the terms, one finds

$$\begin{aligned} f &= \left(\mu_1 dx^{d+1} + \tau dz^1\right) \wedge \theta_1 + \tau^{-1}\left(\tau dx^{d+1} - \mu_1 dz^1\right) \wedge \sum_{j>1} \mu_j \theta_j \\ &\quad + \sum_{j>1, A>1} \chi_j^A dz^A \wedge \theta_j + \Phi \ . \end{aligned} \tag{2.54}$$

Writing

$$\begin{aligned} f_1 &= \left(\mu_1 dx^{d+1} + \tau dz^1\right) \wedge \theta_1 \\ \tilde{f} &= \tau^{-1}\left(\tau dx^{d+1} - \mu_1 dz^1\right) \wedge \sum_{j>1} \mu_j \theta_j + \sum_{j>1, A>1} \chi_j^A dz^A \wedge \theta_j + \Phi \end{aligned} \tag{2.55}$$

we have found $f = f_1 + \tilde{f}$ where, as a consequence of (2.50) and (2.52), it follows that Φ has no components in the z^1 direction.

So, in both cases, we have the decomposition

$$f = f_1 + \tilde{f} \tag{2.56}$$

where f_1 is a simple 4-form, and f_1, $\tilde{f}$ are totally orthogonal i.e. $f_1^{\mu_1\mu_2\mu_3\nu}\tilde{f}^{\mu_4\mu_5\mu_6}{}_\nu = 0$.

Having obtained this result, it is straightforward to prove that if such an f satisfies (1.2), then

$$f = \sum_{s=1}^{N} f_s \tag{2.57}$$

where f_s are totally orthogonal simple 4-forms. The proof proceeds by induction on the spacetime dimension D ($D \geq 4$). The result is clearly true for $D = 4$. Suppose it is true for $4 \leq D \leq d$. Suppose that $D = d + 1$. Then by the previous reasoning, one has the decomposition $f = f_1 + \tilde{f}$, where f_1 is a simple 4-form, and $f_1, \tilde{f}$ are totally orthogonal. It follows that $\tilde{f}$ must satisfy (1.2). Then either $\tilde{f} = 0$ and we are done, or $\tilde{f}$ is a nonzero 4-form in dimension $d - 3$, in which case it follows that one can decompose $\tilde{f}$ into a finite sum of orthogonal simple 4-forms, each of which is also orthogonal to f_1.

Hence we conclude that the decomposition (2.57) holds for all 4-forms f satisfying (1.2).

3. Discussion

Given the results presented here, the maximally supersymmetric field theory Lagrangian based on the four-dimensional algebra with $f^{\mu_1\mu_2\mu_3\mu_4} = \epsilon^{\mu_1\mu_2\mu_3\mu_4}$ is rather enigmatic. If it is not to be an isolated curiosity, the assumptions going into the general constructions of [3 – 5] need to be relaxed. One possibility is to relax the condition that the metric living on the algebra is positive definite and some discussion recently appeared in [16]. A different possibility is to not demand a Lagrangian description, but to work instead at the level of the field equations and this was recently discussed in [15]. Another possibility, which also does not use totally antisymmetric structure constants, was considered in [12].

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References

[1] J.H. Schwarz, *Superconformal Chern-Simons theories*, *JHEP* **11** (2004) 078 [hep-th/0411077].

[2] A. Basu and J.A. Harvey, *The M2-M5 brane system and a generalized Nahm's equation*, *Nucl. Phys.* **B 713** (2005) 136 [hep-th/0412310].

[3] J. Bagger and N. Lambert, *Modeling multiple M2's*, *Phys. Rev.* **D 75** (2007) 045020 [hep-th/0611108].

[4] A. Gustavsson, *Algebraic structures on parallel M2-branes*, arXiv:0709.1260.

[5] J. Bagger and N. Lambert, *Gauge symmetry and supersymmetry of multiple M2-branes*, *Phys. Rev.* **D 77** (2008) 065008 [arXiv:0711.0955].

[6] J. Bagger and N. Lambert, *Comments on multiple M2-branes*, *JHEP* **02** (2008) 105 [arXiv:0712.3738].

[7] A. Gustavsson, *Selfdual strings and loop space Nahm equations*, *JHEP* **04** (2008) 083 [arXiv:0802.3456].

[8] S. Mukhi and C. Papageorgakis, *M2 to D2*, *JHEP* **05** (2008) 085 [arXiv:0803.3218].

[9] M.A. Bandres, A.E. Lipstein and J.H. Schwarz, *$N = 8$ superconformal Chern-Simons theories*, *JHEP* **05** (2008) 025 [arXiv:0803.3242].

[10] D.S. Berman, L.C. Tadrowski and D.C. Thompson, *Aspects of multiple membranes*, arXiv:0803.3611.

[11] M. Van Raamsdonk, *Comments on the Bagger-Lambert theory and multiple M2-branes*, *JHEP* **05** (2008) 105 [arXiv:0803.3803].

[12] A. Morozov, *On the problem of multiple M2 branes*, *JHEP* **05** (2008) 076 [arXiv:0804.0913].

[13] N. Lambert and D. Tong, *Membranes on an orbifold*, arXiv:0804.1114.

[14] J. Distler, S. Mukhi, C. Papageorgakis and M. Van Raamsdonk, *M2-branes on M-folds*, *JHEP* **05** (2008) 038 [arXiv:0804.1256].

[15] U. Gran, B.E.W. Nilsson and C. Petersson, *On relating multiple M2 and D2-branes*, arXiv:0804.1784.

[16] P.-M. Ho, R.-C. Hou and Y. Matsuo, *Lie 3-algebra and multiple M2-branes*, arXiv:0804.2110.

[17] J. Gomis, A.J. Salim and F. Passerini, *Matrix theory of type IIB plane wave from membranes*, arXiv:0804.2186.

[18] E.A. Bergshoeff, M. de Roo and O. Hohm, *Multiple M2-branes and the embedding tensor*, arXiv:0804.2201.

[19] K. Hosomichi, K.-M. Lee and S. Lee, *Mass-deformed Bagger-Lambert theory and its BPS objects*, arXiv:0804.2519.

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[20] J. Figueroa-O'Farrill and G. Papadopoulos, *Pluecker-type relations for orthogonal planes*, *J. Geom. Phys.* **49** (2004) 294 [math.DG/0211170].

[21] G. Papadopoulos, *M2-branes, 3-Lie algebras and Plucker relations*, *JHEP* **05** (2008) 054 [arXiv:0804.2662].

[22] P.-A. Nagy, *Prolongations of Lie algebras and applications*, arXiv:0712.1398.